数理解析研究所講究録
第 874 巻 1994 年 180-191

Information Dynamics and Its Applications

Masanori OHYA

Department Information Sciences
Science University of Tokyo
Noda City, Chiba 278, JAPAN

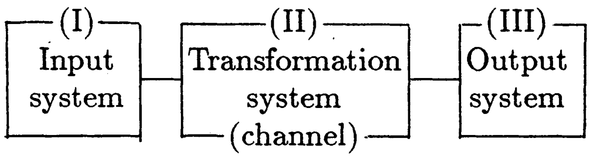
§1 What is Information Dynamics

Information dynamics was proposed in [O.4] to study several dynamical systems with complexity. Information dynamics is a syntheis of the dynamics of state change and the theory of complexity.

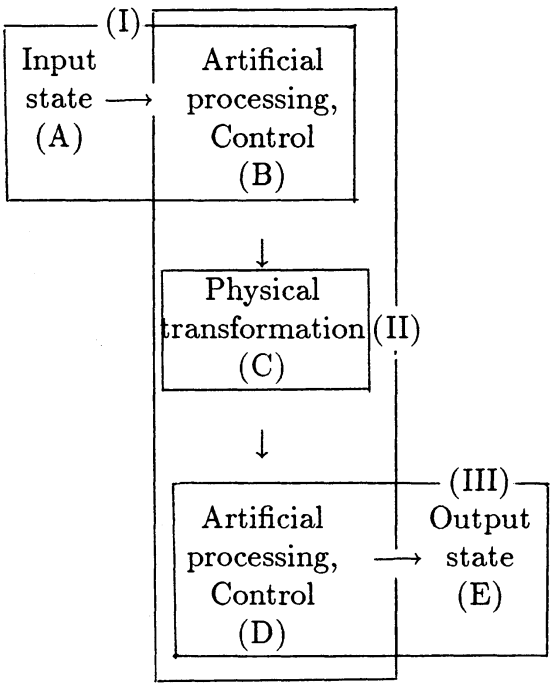
1.1 Dynamics

Almost all systems are described by states and their dynamics are considered as the state change.

Symbolically, almost all systems are divided into three parts such as



(chart 1)



(Chart 2)

In the above chart 2, (I) (= (A) + (B)) corresponds to an input system, (II) (= (B) + (C) + (D)) does to a transformation (channel) system (III) (= (D) + (E)) to an output system. More complex systems are constructed from this first structure.

A "naked" state (A) is artificially processed or controlled to a "dressed" state, and it is suffered to change by a physical (natural) transformation, and it is again artificially processed and controlled. The fundamental part of the process for this state change is obviously "A → C → E".

1.2 The outline of state change without mathematical details

Let an input dynamical system and an output dynamical system be described by $(\mathcal{A}, \mathfrak{S}, \alpha)$ and $(\overline{\mathcal{A}}, \overline{\mathfrak{S}}, \overline{\alpha})$, respectively. Here $\mathcal{A}$ is a set of all objects to be observed and $\mathfrak{S}$ is a set of all means getting the observed value for each element A in $\mathcal{A}$, and α describes an inner evolution of the input system. We call $\mathfrak{S}$ a "state space" here. Same for the output system $(\overline{\mathcal{A}}, \overline{\mathfrak{S}}, \overline{\alpha})$. Thus we may say

[Giving a mathematical structure to input and output triples $\equiv$ Having a theory].

For instance, to non-mathematical frame $(\mathcal{A}, \mathfrak{S}, \alpha(G))$, a speculation containing classical system and quantum system is the C*-system.

Let $(\mathcal{A}, \mathfrak{S}, \alpha)$ input system with $I \in \mathcal{A}$ and $(\overline{\mathcal{A}}, \overline{\mathfrak{S}}, \overline{\alpha})$ output system with $\overline{I} \in \overline{\mathcal{A}}$.

A map $\Lambda^* : \mathfrak{S} \to \overline{\mathfrak{S}}$ is called a channel, which gives us a bridge between two systems. Let us consider a subset $\mathcal{S}$ of $\mathfrak{S}$, called reference system, in which we can perform the observation.

$$\begin{array}{ccc} & \Lambda^* & \\ \mathcal{S}(\mathcal{A}) \to & \boxed{\text{channel}} & \to \mathcal{S}(\overline{\mathcal{A}}) \\ & \wr\ \text{example} & \\ \mathcal{S}(\mathcal{A}) \otimes \mathcal{S}(\mathcal{B}) & \xrightarrow{\Lambda^*} & \mathcal{S}(\overline{\mathcal{A}}) \otimes \mathcal{S}(\overline{\mathcal{B}}) \end{array}$$

1.3 Examples

(1) Causal system:

(I) ... $x \in R^n$.
(II) ... $\dot{x} = f(x)$ or Φ_t generated f.
(III)... $x(t) \equiv \Phi_t(x(0))$, Φ_t : generated by f.

Remark: If $x(t)$ can not be obtained directly, then examines the properties of f and find some proper approximation to obtain Φ_t.
$\Longrightarrow$ Chaotic system

(2) Signal transmission:

(I) ... causal signal $x(t) \Longrightarrow$ coding x_n.
(by e.g., Shannon's sampling theorem, i.e, Fourier transformation + cut off)
(II) ... some transformation $y_n = f(x_n)$.
(III)... Interpolation (Inverse Fourier transformation)
$y_n \to y(t)$.

(3) Discrete systems:

(I) ... input probability distribution
$p = \{p_1, \ldots, p_n\}$ of events $X = \{x_1, \ldots, x_n\}$.
(II) ... transition probability $(p(i|j))$
(III)... output probability distribution
$q = \{q_i\}; q_i = \sum_j p(i|j)p_j$.

(4) Continuous systems:

(I) $\cdots$ probability measure μ on measurable space $(\Omega, \mathfrak{F})$.
(II) $\cdots$ Markov kernel
$\lambda : \Omega \times \overline{\mathfrak{F}} \to [0,1]$ s.t.
$\lambda(x, \cdot) \in P(\overline{\Omega})$ and $\lambda(\cdot, A) \in M(\Omega)$.
(III) $\cdots$ $\overline{\mu} \equiv \int_\Omega \lambda(x, \cdot) d\mu$

(5) Quantum system 1:

(I) $\cdots$ $x \in \mathcal{H}$ (Hilbert space)
(II) $\cdots$ unitary group U or semigroup V etc.
(III) $\cdots$ $y = Ux$ or $Vx \in \mathcal{H}$.

(6) Quantum system 2:

(I) $\cdots$ density operator $\rho \in T(\mathcal{H})_{+,1}$.
(II) $\cdots$ $\Lambda^* \equiv \mathrm{Ad}U$ or $\mathrm{Ad}V$.
(III) $\cdots$ $\overline{\rho} = \Lambda^* \rho$.

(7) C*-system containing all above.

(I) $\cdots$ $(\mathcal{A}, \mathfrak{S}, \alpha(G))$ input system.
$(\overline{\mathcal{A}}, \overline{\mathfrak{S}}, \overline{\alpha}(\overline{G}))$ output system.
(II) $\cdots$ $\Lambda^* : \mathfrak{S}(\mathcal{A}) \to \overline{\mathfrak{S}}(\overline{\mathcal{A}})$ a dual map of completely positive map.
(III) $\cdots$ $\overline{\varphi} = \Lambda^* \varphi$.

1.4 Complexities of a state (or system) φ

Another speculation of information dynamics is complexity associated with two systems. The complexities satisfy the following properties:

Let $(\mathcal{A}_1, \mathfrak{S}_1, \alpha^1(G))$, $(\mathcal{A}_2, \mathfrak{S}_2, \alpha^2(G))$ be two systems and $(\mathcal{A}, \mathfrak{S}, \alpha(G))$ be the compound system such that $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, and Λ^* be a channel from $\mathfrak{S}_1$ to $\mathfrak{S}_2$.

(i) For any state $\varphi \in \mathcal{S} \subset \mathfrak{S}_1$, $\mathrm{C}^{\mathcal{S}}(\varphi) \geq 0$ and $\mathrm{T}^{\mathcal{S}}(\varphi; \Lambda^*) \geq 0$.

(ii) For any bijection j from $ex\mathfrak{S}_1$ to $ex\mathfrak{S}_2$, (or $\exists$ bijection $j : \mathfrak{S} \to \mathfrak{S}$)

$$\mathrm{C}^{j(\mathcal{S})}(j(\varphi)) = \mathrm{C}^{\mathcal{S}}(\varphi)$$
$$\mathrm{T}^{j(\mathcal{S})}(j(\varphi); \Lambda^*) = \mathrm{T}^{\mathcal{S}}(\varphi; \Lambda^*).$$

(iii) For any $\Phi \in \mathcal{S} \subset \mathfrak{S}$, put $\varphi \equiv \Phi \upharpoonright \mathcal{A}_1 \in \mathcal{S}_1 \equiv \mathcal{S} \upharpoonright \mathcal{A}_1$ and $\psi \equiv \Phi \upharpoonright \mathcal{A}_2 \in \mathcal{S}_2 \equiv \mathcal{S} \upharpoonright \mathcal{A}_2$

$$\mathrm{C}^{\mathcal{S}}(\Phi) \leq \mathrm{C}_1^{\mathcal{S}}(\varphi) + \mathrm{C}_2^{\mathcal{S}}(\psi).$$

Moreover
(iv) $0 \leq \mathrm{T}^{\mathcal{S}}(\varphi; \Lambda^*) \leq \mathrm{C}^{\mathcal{S}}(\varphi)$.
(v) $\mathrm{T}^{\mathcal{S}}(\varphi; id) = \mathrm{C}^{\mathcal{S}}(\varphi)$, where "$id$" is an identity channel from $\mathfrak{S}_1$ to $\mathfrak{S}_1$; $id(\varphi) = \varphi$, $\forall \varphi \in \mathfrak{S}_1$.

(Example)

(1) $\mathrm{C}^{\mathcal{S}}(\varphi) = \mathrm{S}^{\mathcal{S}}(\varphi)$; entropy of $\varphi \in \mathcal{S}$.
$\mathrm{T}^{\mathcal{S}}(\varphi; \Lambda^*) = \mathrm{I}^{\mathcal{S}}(\varphi; \Lambda^*)$; mutual entropy.

(2) Evolution measure for genes.

(3) Some fractal dimensions for a state φ.

(4) Chaitin's complexity of a sequencial state.

The information dymanics is described by

$(\mathcal{A}, \mathfrak{S}, \alpha(G); \overline{\mathcal{A}}, \overline{\mathfrak{S}}, \overline{\alpha}(\overline{\mathbf{G}}); \Lambda^*; \mathbf{C}^{\mathcal{S}}(\varphi), \mathbf{T}^{\mathcal{S}}(\varphi; \Lambda^*))$ **and some other functions (relations) R.**

Therefore, for each system of interest, we
(i) mathematically determine
$\mathcal{A}, \mathfrak{S}, \alpha(G); \overline{\mathcal{A}}, \overline{\mathfrak{S}}, \overline{\alpha}(\overline{G})$,
(ii) choose Λ^* and R, and
(iii) define $\mathrm{C}^{\mathcal{S}}(\varphi), \mathrm{T}^{\mathcal{S}}(\varphi; \Lambda^*)$.

Once we set the above (i) $\sim$ (iii) in general quantum system (GQS), then our theory contains almost all (scientific) systems and we apply this general frames to the following topics.

(1) Study of optical communication processes.
(2) Formulation of fractal dimensions of states, and study of complexity for some sequences.
(3) Define genetic matrix and costruction of phylogenetic tree of evolution of species.
(4) Study of recognition processes.

In the next section, we a bit more explain two of the above fundamental objects in GQS, channel Λ^* and complexities C, T. In §3, we apply general frameworks to formulate the fractal dimensions of states.

§2 Channel and Complexities

2.1 Channel and Lifting [A.1]

Let $(\mathcal{A}, \mathfrak{S}, \alpha)$ be an input system and $(\overline{\mathcal{A}}, \overline{\mathfrak{S}}, \overline{\alpha})$ be an output system. Very often $\mathcal{A} = \overline{\mathcal{A}}$, $\mathfrak{S} = \overline{\mathfrak{S}}$, $\alpha = \overline{\alpha}$.

A map $\Lambda^* : \mathfrak{S} \to \overline{\mathfrak{S}}$ is called "a channel". Furthermore

(1) Λ^* is a "linear channel"if Λ^* is affine.
(2) Λ^* is a "completely positive channel"if $\Lambda : \overline{\mathcal{A}} \to \mathcal{A}$ satisfies

$$\sum_{i,j=1}^{n} B_i^* \Lambda(A_i^* A_j) B_j \geq 0$$

for any $n \in \mathbf{N}$, any $B_j \in \mathcal{A}$ and any $A_i \in \overline{\mathcal{A}}$ A "lifting"from $\mathcal{A}$ to $\mathcal{A} \otimes \overline{\mathcal{A}}$ is a continuous map such as

$$\mathcal{E}^* : \mathfrak{S}(\mathcal{A}) \to \mathfrak{S}(\mathcal{A} \otimes \overline{\mathcal{A}})$$

A lifting $\mathcal{E}^*$ is said to be "nondemolition"for a state φ if

$$(\mathcal{E}^*\varphi)(A \otimes I) = \varphi(A) \text{ for } \forall A \in \mathcal{A}.$$

Given a lifting $\mathcal{E}^*$, we can construct channels
$\Lambda^* : \mathfrak{S} \to \overline{\mathfrak{S}}$
by $\Lambda^*\varphi(\overline{\mathcal{A}})=(\mathcal{E}^*\varphi)(I \otimes \overline{A}), \forall \overline{A} \in \overline{\mathcal{A}}$
$\overline{\Lambda}^* : \mathfrak{S} \to \mathfrak{S}$
by $\overline{\Lambda}^*\varphi(\mathcal{A}) = (\mathcal{E}^*\varphi)(A \otimes I), \forall A \in \mathcal{A}$

Given a channel $\Lambda^* : \mathfrak{S} \to \overline{\mathfrak{S}}$ and $\varphi \in \mathcal{S} \subset \mathfrak{S}$, take a certain decomposition such that

$$\varphi = \int_S \omega d\mu$$

Then

$$\mathcal{E}^*\varphi = \int_S \omega \otimes \Lambda^* \omega d\mu \quad \text{(compound state)}$$

is a nonlinear nondemolition lifting.

Examples of channels and liftings

Let be $\rho = \sum \lambda_n \rho_n$ be a certain decomposition.

(1) Unitary evolution:

$$\rho \to \Lambda_t^* \rho \equiv U_t^* \rho U_t, t \in R \Rightarrow \mathcal{E}^* \rho = \sum \lambda_n \rho_n \otimes \Lambda^* \rho_n$$

where U_t is a unitary operator $U_t = \exp(\mathrm{itH})$

(2) Semigroup evolution:

$$\rho \to \Lambda_t^*\rho \equiv V_t^*\rho V_t \Rightarrow \mathcal{E}^*\rho = \sum \lambda_n\rho_n \otimes \Lambda^*\rho_n$$

where $\{V_t;\ t\in R^+\}$ is a one parameter semigroup on $\mathcal{H}$.

(3) Measurement: Measure A $= \sum_n a_n P_n$ (spectral decomposition) in a state ρ,

$$\rho \to \Lambda^*\rho = \sum_n P_n\rho P_n \Rightarrow \mathcal{E}^*\rho = \sum \lambda_n\rho_n \otimes \Lambda^*\rho_n.$$

(4) Reduction (Open system dynamics):

$$\begin{array}{ccc} \mathfrak{S}(\mathcal{H}) & \xrightarrow{\quad\text{interaction}\quad} & \mathfrak{S}(\mathcal{K}) \\ \Cup & & \Cup \\ \rho & & \sigma \end{array}$$

$$\mathcal{E}_t^*\rho \equiv \theta_t = U_t^*(\rho\otimes\sigma)U_t,$$
$$\rho \to \Lambda_t^*\rho = tr_{\mathcal{K}}\theta_t.$$

For (3) and (4), $\mathcal{A}$: observables of a system, $\overline{\mathcal{A}}$: observables of a apparatus. Given $\mathcal{E}^*$ (interaction between two systems) $\Longrightarrow$ Λ^* and $\overline{\Lambda^*}$ are obtained $\Longrightarrow$ $\overline{\Lambda^*\varphi}$; state of the system; $\Lambda^*\varphi$; state of the apparatus.

(5) Isometric lifting

$$V : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_2 \text{ isometry } (V^*V = I_{\mathcal{H}_1})$$

$$\mathcal{E}^*\rho = V\rho V^*, \forall\rho \in \mathfrak{S}(\mathcal{H}_1)$$

(6) Optical communication channel [O.1, O.2, O.8]:

$$\begin{array}{ccc} \text{(Noise) } \mathfrak{S}(\mathcal{K}_1)\text{(density operators on } \mathcal{K}_1) & & \\ \downarrow & & \\ \mathfrak{S}(\mathcal{H}_1) \ni \rho & \xrightarrow{\qquad\qquad} & \bar{\rho} = \Lambda^*\rho \in \mathfrak{S}(\mathcal{H}_2) \\ & & \downarrow \\ & & \text{(Loss) } \mathfrak{S}(\mathcal{K}_2) \end{array}$$

Let $\nu \in \mathfrak{S}(\mathcal{K}_1)$ be a state representing the noise and a, π, γ be the following maps: (1) a : B$(\mathcal{H}_2) \to$ B$(\mathcal{H}_2 \otimes \mathcal{K}_2)$ given by a(A) = A$\otimes$I for any A $\in$ B$(\mathcal{H}_2)$, (2) π :

$B(\mathcal{H}_2 \otimes \mathcal{K}_2) \to B(\mathcal{H}_1 \otimes \mathcal{K}_1)$ completely positive with $\pi(I) = I$, (3) $\gamma : B(\mathcal{H}_1 \otimes \mathcal{K}_1) \to B(\mathcal{H}_1)$ by $\gamma(Q) = tr_{\mathcal{K}_1} \nu Q$ for any $Q \in B(\mathcal{H}_1 \otimes \mathcal{K}_1)$.

$$\Lambda = \gamma \circ \pi \circ a.$$

Then

$$\Lambda^* = a^* \circ \pi^* \circ \gamma^*$$

or equivalently,

$$\boxed{\Lambda^* \rho = tr_{\mathcal{K}_2} \pi^*(\rho \otimes \nu), \forall \rho \in \mathfrak{S}(\mathcal{H}_1).}$$

When $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}$,

$$\mathcal{E}^* : \rho \in \mathfrak{S}(\mathcal{H}) \to \pi^*(\rho \otimes \nu) \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$$

is a lifting, and

$$\boxed{\Lambda^* \rho = tr_{\mathcal{K}} \mathcal{E}^* \rho.}$$

2.2 Examples of Complexities

Let $(\mathcal{A}, \mathfrak{S}, \alpha(R))$ be a C*-dynamical system and $\mathcal{S}$ be a weak* compact and convex subset of $\mathfrak{S}$. Take a decomposition of φ such that

$$\varphi = \int_{\mathcal{S}} \omega d\mu$$

and let $M^{\mathcal{S}}(\varphi)$ be the set of all decomposition measures μ of φ. Set

$$T_\mu^{\mathcal{S}}(\varphi; \Lambda^*) = \int_{\mathcal{S}} S(\Lambda^* \omega | \Lambda^* \varphi) d\mu$$

where $S(\cdot|\cdot)$ is the relative entropy. Then the examples of two complexities are defined by

$$T^{\mathcal{S}}(\varphi; \Lambda^*) = \sup\{ \int_{\mathcal{S}} S(\Lambda^* \omega | \Lambda^* \varphi) d\mu; \ \mu \in M^{\mathcal{S}}(\varphi)\}$$
$$C^{\mathcal{S}}(\varphi) = T^{\mathcal{S}}(\varphi; id)$$

The above id is an identity channel from $\mathfrak{S}$ to $\mathfrak{S}$.

In particular, let $\mathcal{A} = B(\mathcal{H})$ and $\mathfrak{S} = T(\mathcal{H})_{+,1}$, $\varphi(\cdot) = tr\rho\cdot$, $\mathcal{S} = \mathfrak{S}$ and $M^{\mathcal{S}}(\varphi)$ is the set of all extremal decompsitions of ρ. Then the entropy $S(\rho)$ and the mutual entropy $I(\rho; \Lambda^*)$ become two complexities ;

$$T(\rho; \Lambda^*) = I(\rho; \Lambda^*)$$
$$C(\rho) = S(\rho) = -tr\rho \log \rho$$

The above mutual entropy is defined by

$$I(\rho; \Lambda^*)(\equiv I(\varphi; \Lambda^*)) = \sup\{I_E(\rho; \Lambda^*); \ E \equiv \{E_n\} \text{ of } \rho\},$$
$$I_E(\rho; \Lambda^*) = S(\sigma_E | \sigma_0) = tr\sigma_E(\log \sigma_E - \log \sigma_0),$$

where $\sigma_0 = \rho \otimes \Lambda^* \rho$, $\sigma_E = \sum_n \lambda_n E_n \otimes \Lambda^* E_n$ and $\rho = \sum_n \lambda_n E_n$ is Schatten decomposition (one dimensional spectral decomposition).

§3 Fractal Dimensions of States [O.5]

Usual fractal theory mostly treats geometrical sets [M.1]. It is desirable to extend the fractal theory so as to be applicable to some other objects. For this purpose, we introduce the fractal dimension for general states. First we recall two fractal dimensions of geometrical sets.

<Scaling dimension >

We observe a complex set F built from a fundamental pattern. If the number of the patterns observed is $N(1)$ when the scale is very rough, say 1, and the number is $N(r)$ when the scale is r, then we call the dimansion defined by

$$d_S(F) = \frac{\log(N(r)/N(1))}{\log(1/r)}$$

the scaling dimension of the set F.

<Capacity dimension >

Let try to cover a set F in the n-dimension Euclidian space $\boldsymbol{R}^{\mathrm{n}}$ by a certain convex set with the diameter ε. If the Smallest number of the convex sets needed to conver the set F is $N(\varepsilon)$, then we call the dimension given by

$$d_C(F) = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}$$

the capacity dimension (or the ε-entropy dimension) of the set F.

These two fractal dimensions become equal for almost all sets in which we can compute these dimensions.

The ε-entropy is extensively studied by Kolmogorov [K.1] and his ε-entropy is for a probability measure, which gives us an idea to define the ε-entropy for a general state.

3.1 ε-entropy in GQS

Kolmogorov introduced the notion of ε-entropy in probability space $(\Omega, \mathcal{F}, \mu)$. His formulation is as follows: For two random variables $f, g \in M(\Omega)$, the mutual entropy $\mathrm{I}(f,g)$ is defined by the joint probability measure $\mu_{f,g}$ and the direct product measure $\mu_f \otimes \mu_g$ such that

$$\mathrm{I}(f,g) = \mathrm{S}(\mu_{f,g}, \mu_f \otimes \mu_g)$$

The ε-entropy for a random variable f is given by

$$\mathrm{S}(f,g) \equiv \inf\{\mathrm{I}(f,g);\ g \in M_d(f,\varepsilon)\}$$

where $M_d(f,\varepsilon) \equiv \{g \in M(\Omega);\ \sqrt{\int_\Omega d(f,g)^2 d\mu} < +\infty\}$ with the distance $d(f,g)$ between f and g.

Let $\mathcal{C}$ be the set of all channels and define two sets;

$$\mathcal{C}_1(\Lambda^*;\varphi) = \{\Gamma^* \in \mathcal{C}; \Gamma^*\varphi = \Lambda^*\varphi\},$$
$$\mathcal{C}_2(\varphi;\varepsilon) = \{\Gamma^* \in \mathcal{C}; \| \varphi - \Gamma^*\varphi \| \leq \varepsilon\},$$

Then the ε-entropy of a state w.r.t. $\mathcal{S}$ is defined by

$$\mathrm{S}^{\mathcal{S}}(\varphi;\varepsilon) = \inf\{\mathrm{J}^{\mathcal{S}}(\varphi;\Lambda^*); \Lambda^* \in \mathcal{C}_2(\varphi;\varepsilon)\},$$

where

$$\mathrm{J}^{\mathcal{S}}(\varphi;\Lambda^*) = \sup\{\mathrm{T}^{\mathcal{S}}(\varphi;\Gamma^*); \Gamma^* \in \mathcal{C}_1(\Lambda^*;\varphi)\}.$$

We denote $\mathrm{S}^{\mathfrak{S}}(\varphi;\varepsilon)$ by $\mathrm{S}(\varphi;\varepsilon)$ in the sequel for simplicity. When T equals to the mutual entropy, the ε-entropy is called a Kolmogorov type.

Theorem

(1) If $\mathcal{A} = \mathrm{C}(\Omega), \varphi = \mu_f$ with a random variable f and T is a classical mutual entropy, then $\mathrm{S}(\varphi;\varepsilon) = \mathrm{S}(f;\varepsilon)$ of Kolmogorov.

(2) $S(\varphi)$ is a complexity and $S(\varphi;\epsilon)$ is a transmitted complexity.

3.2 Fractal Dimensions in GQS

The capacity dimension of a state φ w.r.t $\mathcal{S}$ is defined by

$$d_c^{\mathcal{S}}(\varphi) \equiv \lim_{\varepsilon\to\infty} d_c^{\mathcal{S}}(\varphi;\varepsilon),$$

where

$$d_c^{\mathcal{S}}(\varphi;\varepsilon) = \frac{\mathrm{S}^{\mathcal{S}}(\varphi;\varepsilon)}{\log(1/\varepsilon)}.$$

$d_c^{\mathcal{S}}(\varphi;\varepsilon)$ is called the capacity dimension of ε-order. The information dimension of a state φ for $\mathcal{S}$ of ε order is defined by

$$d_{\mathrm{I}}^{\mathcal{S}}(\varphi;\varepsilon) = \frac{\mathrm{S}^{\mathcal{S}}(\varphi;\varepsilon)}{\mathrm{I}(\varepsilon)},$$

where $\mathrm{I}(\varepsilon)$ is a certain normalization constant function in the limit $\varepsilon \to 0$ such as

$$\lim_{\varepsilon\to 0} \mathrm{I}(\varepsilon) = \mathrm{S}^{\mathcal{S}}(\varphi)$$

Some Applications

In a classical system, we put

$$\alpha(P\,;\,\varepsilon) \equiv \frac{\mathrm{S}(P;\varepsilon)}{\log(\frac{1}{\varepsilon})},$$
$$\alpha(P) \equiv \lim_{\varepsilon\to 0} \alpha(P\,;\,\varepsilon)$$
$$\beta(P\,;\,\varepsilon) \equiv \frac{\mathrm{S}(P;\varepsilon)}{\mathrm{S}(P)} \qquad (\mathrm{S}(P) = \mathrm{C}(P))$$

for a probability distribution P. Some computations of the fractal dimensions α and β are listed below.

We consider the case consisting of four events such as DNA sequences, say $P = \{p_k\ ;\ k = 1, 2, 3, 4\}$, in which ε is $\frac{1}{4}$ but can not be 0.

State P	S(P)	S(P;$\frac{1}{4}$)	β (P;$\frac{1}{4}$)
$(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$	1.3863	1.1476	0.8278
$(\frac{3}{8},\frac{1}{4},\frac{1}{4},\frac{1}{8})$	1.3209	1.0397	0.7871
$(\frac{3}{8},\frac{3}{8},\frac{1}{8},\frac{1}{8})$	1.2555	0.9743	0.7760
$(\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{8})$	1.2130	0.9003	0.7422
$(\frac{3}{8},\frac{3}{8},\frac{1}{4},0)$	1.0822	0.8010	0.7402
$(\frac{5}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8})$	1.0735	0.7356	0.6852
$(\frac{1}{2},\frac{1}{4},\frac{1}{4},0)$	1.0397	0.7270	0.6992
$(\frac{1}{2},\frac{3}{8},\frac{1}{8},0)$	0.9743	0.6616	0.6791
$(\frac{5}{8},\frac{1}{4},\frac{1}{8},0)$	0.9003	0.5623	0.6246
$(\frac{3}{4},\frac{1}{8},\frac{1}{8},0)$	0.7356	0.3768	0.5122
$(\frac{1}{2},\frac{1}{2},0,0)$	0.6931	0.3804	0.5488
$(\frac{5}{8},\frac{3}{8},0,0)$	0.6616	0.3236	0.4891
$(\frac{3}{4},\frac{1}{4},0,0)$	0.5623	0.2035	0.3619
$(\frac{7}{8},\frac{1}{8},0,0)$	0.3768	0	0

(m=4,n=19)

$$P = \{\frac{6}{19}, \frac{6}{19}, \frac{6}{19}, \frac{1}{19}\} \quad Q = \{\frac{9}{19}, \frac{4}{19}, \frac{4}{19}, \frac{2}{19}\}$$

$S(P) =$	1.2470	$S(Q) =$	1.2470
$\beta(P; 2/n) =$	0.8788	$\beta(Q; 2/n) =$	0.8628

(m=4,n=20)

$$P = \{\frac{8}{20}, \frac{6}{20}, \frac{6}{20}, \frac{0}{20}\} \quad Q = \{\frac{12}{20}, \frac{4}{20}, \frac{2}{20}, \frac{2}{20}\}$$

1.0889	1.0889
0.8558	0.8381

$$P = \{\frac{9}{20}, \frac{8}{20}, \frac{2}{20}, \frac{1}{20}\} \quad Q = \{\frac{12}{20}, \frac{3}{20}, \frac{3}{20}, \frac{2}{20}\}$$

1.1059	1.1059
0.8530	0.8406

(m=4,n=21)

$$P = \{\frac{9}{21}, \frac{8}{21}, \frac{3}{21}, \frac{1}{21}\} \quad Q = \{\frac{12}{21}, \frac{3}{21}, \frac{3}{21}, \frac{3}{21}\}$$

1.1537	1.1537
0.8658	0.8545

(m=4,n=22)

$$P = \{\frac{9}{22}, \frac{8}{22}, \frac{4}{22}, \frac{1}{22}\} \quad Q = \{\frac{12}{22}, \frac{4}{22}, \frac{3}{22}, \frac{3}{22}\}$$

1.1840	1.1840
0.8752	0.8647

(m=4,n=23)

$$P = \{\frac{9}{23}, \frac{8}{23}, \frac{5}{23}, \frac{1}{23}\} \quad Q = \{\frac{12}{23}, \frac{5}{23}, \frac{3}{23}, \frac{3}{23}\}$$

1.1537	1.1537
0.8658	0.8545

(m=4,n=24)

$$P = \{\frac{10}{24}, \frac{9}{24}, \frac{5}{24}, \frac{0}{24}\} \quad Q = \{\frac{15}{24}, \frac{4}{24}, \frac{3}{24}, \frac{2}{24}\}$$

1.1537	1.1537
0.8658	0.8545

Finally, we discuss a noncommutative case. For a density operator ρ and $\rho' = U\rho U^*$ transformed by a unitary operator U, then $S(\rho) = S(\rho')$ but $S(\rho\,;\,\varepsilon) \neq S(\rho'\,;\,\varepsilon)$, hence the fractal dimensions with ε-order of ρ and ρ' are different.

For instance, take

$$\rho = \begin{pmatrix} 0.75 & 0.001 \\ 0.001 & 0.75 \end{pmatrix}, \quad \rho' = \begin{pmatrix} 0.75002 & 0 \\ 0 & 0.24998 \end{pmatrix}$$

then

$$S(\rho) = S(\rho') = 0.562323295$$
$$S(\rho\,;\,0.01) = 0.11130193$$
$$S(\rho'\,;\,0.01) = 0.11112695$$

These discussions show that the fractal dimension of a state is a certain expression of the complexity, different from that of entropy.

Information dynamics can be applied to quantum communication [O.4, O.6] and genetics [O.3, O.9].

References

[A.1] Accardi, L., Ohya, M. (1992): "Compound channels, transition expectations and liftings", to appear in J. Multivariate Analysis.
[B.1] Benjaballah, C., Hirota, O. and Ohya, M.: "Quantum Optical Communication", to be published in Springer-Verlag.
[K.1] Kolmogorov, A.N. (1963): "Theory of transmission of information", Amer. Math. Soc. Translation, Ser.2, 33, 291
[M.1] Mandelbrot, B.B. (1982): "The Fractal Geometry of Nature", W.H. Freemann and Company, San Fancisco.
[O.1] Ohya, M (1983): "On compound state and mutual information in quantum information theory", IEEE Trans. Information Theory, **29**, pp. 770–777.
[O.2] Ohya, M. (1989): "Some aspects of quantum information theory and their applications to irreversible processes", Rep. Math. Phys. **27**, pp. 19–47.
[O.3] Ohya, M. (1989): "Information theoretical treatment of genes", Trans. IEICE, E70, No.5, pp. 556-560.
[O.4] Ohya, M. (1991): "Information dynamics and its application to optical communication processes", in *Quantum Aspects of Optical Communication* ed. by C. Benjaballah, O. Hirota, S. Reynaud (Lecture Notes in Physics **378**, Springer), pp. 81–92.
[O.5] Ohya, M. (1991): "Fractal dimensions of states", in *Quantum Probability and Related Topics VI* (World Scientific, Singapore) pp. 359–369.
[O.6] Ohya, M., Watanabe, N. (1993): "Information dynamics and its application to Gaussian communication processes", Maximum Entropy and Bayesian Methods, Vol.12, pp. 195–203.
[O.7] Ohya, M., Petz, D. (1993): "Quantum Entropy and Its Use", Springer-Verlag.
[O.8] Bendjaballah, C., Hirota, O. and Ohya, M.: "Quantum Optical Communication", to be published in Springer-Verlag.
[O.9] Ohya, M.: in preparation.